Order-Types, Trees, and a

Problem of Erdøs and Hajnal

by

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§0. Introduction

We work in ZFC set theory throughout, and use the usual notation and conventions. In particular, an ordinal is the set of all its predecessors and a cardinal is an initial ordinal. The cardinality of a set X is denoted by $|X|$. As usual, if σ, ρ are order types, $\sigma \leq \rho$ denotes that there is an order preserving monomorphism $F: \sigma \to \rho$. (We use σ, ρ to denote order types, all other lower case Greek letters denoting ordinals, with $\varkappa$, λ, in particular, for cardinals). We assume some familiarity with the notion of constructibility, as in, say, [1]. In particular, we assume the reader knows the usual proof that $V = L \to GCH$ (see [1].) We use L_α, $\alpha \in On$, to denote the levels in the constructible hierarchy.

Let Φ denote the following proposition: Whenever ρ is an order type of cardinality ω_2 such that $\omega_2 \not\leq \rho$ and $\omega_2^* \not\leq \rho$ there is $\sigma \subset \rho$ of cardinality ω_1 such that $\omega_1 \not\leq \sigma$ and $\omega_1^* \not\leq \sigma$. Without assuming GCH, the status of Φ is of little interest, of course, since a set of reals of cardinality ω_2 embeds neither ω_2, ω_2^* nor ω_1, ω_1^*. In [2] (§ 7, Problem I), Erdøs and Hajnal ask what happens to Φ if we do assume GCH. In this paper, we prove that if $V = L$, then $\neg\Phi$. The proof uses Jensen's answer to an earlier question of ours concerning subtrees of ω_1-trees, and we end the paper with a brief discussion of this topic, and a remark concerning the consistency of Φ.

§1. $V = L \to \neg\Phi$

A <u>tree</u> is a poset $\underset{\sim}{T} = \langle T, \leq_T \rangle$ such that for any $x \in T$, $\{y \in T \mid y <_T x\}$ is well ordered by $\leq_T$. (For such $\underset{\sim}{T}$, x, we call the order type of this set (under $<_T$) the <u>height</u> of x in $\underset{\sim}{T}$.) For any ordinal α, we set $T_\alpha = \{x \in T \mid x$ has height α in $\underset{\sim}{T}\}$, the α'th <u>level</u> of $\underset{\sim}{T}$. We write $\underset{\sim}{T} \restriction \alpha$ for $\langle \bigcup_{\beta<\alpha} T_\beta, \leq_T \cap (\bigcup_{\beta<\alpha} T_\beta)^2 \rangle$. A <u>branch</u> of $\underset{\sim}{T}$ is a maximal linearly ordered subset of T. If it has order type α it is an α-<u>branch</u>.

Let θ be an ordinal, κ a cardinal. $\underset{\sim}{T}$ is a (θ,κ)-<u>tree</u> iff $(\forall \alpha < \theta)(T_\alpha \neq \emptyset \ \& \ |T_\alpha| < \kappa) \ \& \ T_\theta = \emptyset$. $\underset{\sim}{T}$ is <u>normal</u> iff whenever $\alpha<\beta<\theta$ and $x \in T_\alpha$ there are distinct $y, y' \in T_\beta$ with $x <_T y, y'$. A tree $\underset{\sim}{T}$ is κ-<u>Aronszajn</u> iff it is a (κ,κ)-tree with no κ-branches. $\underset{\sim}{T}$ is κ-<u>Kurepa</u> iff it is a (κ, κ)-tree with at least κ^+ κ-branches. In both cases, we do not bother to mention the κ if $\kappa = \omega_1$. (We ignore $\kappa = \omega$ from the outset.)

Solovay has proved that if $V = L$, there is a Kurepa tree. For our purposes, a somewhat stronger result is required.

Let us begin by proving three fundamental lemmas on constructibility.

Lemma 1

Assume $V = L$. If $M \prec L_{\omega_1}$, then $M = L_\alpha$ for some $\alpha \leq \omega_1$.

<u>Proof</u>: By absoluteness considerations, it suffices to show that M is transitive. Let $x \in L_{\omega_1}$. Then x is countable. Now, L_{ω_1} has an L_{ω_1}-definable well ordering, $<_L$. Let j_x be the $<_L$-least bijection $j_x : \omega \leftrightarrow x$. Then $j_x \in L_{\omega_1}$ and is L_{ω_1}-definable from x. Hence, $x \in M \to j_x \in M$. But clearly, $\omega \subset M$. Thus $x \in M \to j_x " \omega \subset M \to x \subset M$, as required.

Lemma 2

Assume $V = L$. If $M \prec L_{\omega_2}$, then $M \cap L_{\omega_1} = L_\alpha$ for some $\alpha \leq \omega_1$.

Proof: L_{ω_1} is L_{ω_2}-definable, so clearly $M \cap L_{\omega_1} \prec L_{\omega_1}$. By lemma 1, $M \cap L_{\omega_1} = L_\alpha$ for some $\alpha \leq \omega_1$.

Lemma 3

Let $\beta > \alpha > \omega$, $p \in L_\beta$. Suppose that $L_{\beta+1} \models$ "α is a regular uncountable cardinal" and that L_β is the smallest $X \prec L_\beta$ such thet $p \in X$ and $X \cap \alpha$ is transitive. Then there is an $L_{\beta+1}$-definable map of ω cofinally into α.

Proof: Let us use the notation $X \prec_{\Sigma_n} L_\beta$ to mean that X is an elementary substructure of $\langle L_\beta, \in\rangle$ when we restrict our attention to the Σ_n formulas of set theory only. For each $n \in \omega$, let X_n be the smallest $X \prec_{\Sigma_n} L_\beta$ such that $p \in X$ and $X \cap \alpha$ is transitive. Thus X_n is L_β-definable. But $L_{\beta+1} \models$ "α is a regular uncountable cardinal", so clearly $X_n \cap \alpha \in \alpha$. Let $\alpha_n = X_n \cap \alpha$. Since $\langle X_n | n < \omega\rangle$ is $L_{\beta+1}$-definable, so is $\langle \alpha_n | n < \omega\rangle$. But $\cup_{n<\omega} X_n = L_\beta$ by assumption on L_β, so $\sup_{n<\omega} \alpha_n = \alpha$, and we are done.

The following theorem was proved by Jensen in answer to an old question of ours.

Theorem 4 (Jensen)

Assume $V = L$. Then there is a Kurepa tree no subset of which is (under the inherited ordering) an Aronszajn tree.

Proof: Define a function $h: \omega_1 \to \omega_1$ by setting $h(\alpha)$ = the least γ such that $L_{\gamma+1} \models$ "α is countable". We define an (ω_1, ω_1)-tree $\underset{\sim}{T}$ by induction on the levels. The elements of $\underset{\sim}{T}$ will be countable ordinals, and we have $\alpha <_T \beta \to \alpha < \beta$.

Each $\underset{\sim}{T}\upharpoonright\alpha$ will be a normal (α, ω_1)-tree.

As we proceed, we simultaneously define a function $f: \omega_1 \to \omega_1$ by setting $f(\alpha)$ = the least γ such that $\alpha, T\upharpoonright\alpha \in L_\gamma \prec L_{\omega_1}$ (by lemma 1).

Set $T_0 = \{0\}$. If T_α is defined, $T_{\alpha+1}$ is the result of appointing (in a minimal way) two new ordinals to succeed each member of T_α. We are left with the definition of T_α when $\lim(\alpha)$ and $\underset{\sim}{T}\upharpoonright\alpha$ is defined, and is a normal (α, ω_1)-tree.

Let $S(\alpha)$ be the set of all $U \subset T\upharpoonright\alpha$ such that:

(i) $U \in \bigcup_{\xi<h(\alpha)} L_\xi$;

(ii) U is a normal (α, ω_1)-tree (under the inherited ordering);

(iii) U is <u>thin</u> in $\underset{\sim}{T}\upharpoonright\alpha$ (i.e. for any $x \in U$ there is a $y \in \underset{\sim}{T}\upharpoonright\alpha - U$ such that $x <_T y$);

(iv) $L_{h(\alpha)} \models$ "U is an Aronszajn tree".

We let T_α consist of (minimally appointed ordinals as) one point extensions of each α-branch b of $\underset{\sim}{T}\upharpoonright\alpha$ such that $b \in L_{f(\alpha)}$ and $b \not\subset U$ for any $U \in S(\alpha)$. By condition (iii) above, it is clear that (since $L_{f(\alpha)} \models$ "α and $S(\alpha)$ are countable") $\underset{\sim}{T}\upharpoonright\alpha + 1$ is a normal $(\alpha + 1, \omega_1)$-tree.

Set $\underset{\sim}{T} = \bigcup_{\alpha<\omega_1}\underset{\sim}{T}\upharpoonright\alpha$, a normal (ω_1, ω_1)-tree.

We prove that $\underset{\sim}{T}$ is Kurepa. Suppose not. Then we may let $\langle c_\alpha | \alpha < \omega_1\rangle$ be the $<_L$-least enumeration of all the ω_1-branches of $\underset{\sim}{T}$. $\underset{\sim}{T}$ is clearly L_{ω_2}-definable, and hence so is $\langle c_\alpha | \alpha < \omega_1\rangle$. Using this fact, we define an ω_1-branch b of $\underset{\sim}{T}$ such that $b \neq c_\alpha$ for all $\alpha < \omega_1$, giving the required result.

Define a chain of submodels $X_0 \prec X_1 \prec \ldots \prec X_\nu \prec \ldots \prec L_{\omega_2}$ $(\nu<\omega_1)$ as follows:

X_0 = the smallest $X \prec L_{\omega_2}$;

$X_{\nu+1}$ = the smallest $X \prec L_{\omega_2}$ such that $X_\nu \cup \{X_\nu\} \subset X$;

$X_\gamma = \cup_{\nu<\gamma} X_\nu$ if $\lim(\gamma)$.

For each $\nu < \omega_1$, let $\pi_\nu : X_\nu \cong L_{\psi(\nu)}$. By lemma 2, $\pi_\nu(\omega_1) = \omega_1 \cap X_\nu = \alpha_\nu$, say. (Note that $\langle \alpha_\nu | \nu < \omega_1 \rangle$ is a continuous sequence). Hence $\pi_\nu(\underset{\sim}{T}) = \underset{\sim}{T} \upharpoonright \alpha_\nu$ and $\pi_\nu(\langle c_\alpha | \alpha < \omega_1 \rangle) = \langle c_\alpha \cap \alpha_\nu | \alpha_\nu \rangle$.

Also, $L_{f(\alpha_\nu)} \models$ "α_ν is countable" whereas $L_{\psi(\nu)} \models$ "$\alpha_\nu = \omega_1$", so $\psi(\nu) < f(\alpha_\nu)$ for each $\nu < \omega_1$. It follows that $\langle c_\alpha \cap \alpha_\nu | \alpha < \alpha_\nu \rangle \in L_{f(\alpha_\nu)}$.

We define, by induction, a sequence $b_0 \subset \ldots \subset b_\nu \subset \ldots (\nu < \omega_1)$ such that, for each $\nu < \omega_1$, b_ν is an α_ν-branch of $\underset{\sim}{T} \upharpoonright \alpha_\nu$, with an extension on level α_ν of $\underset{\sim}{T}$, such that $\alpha < \alpha_\nu \to b_\nu \neq c_\alpha \cap \alpha_\nu$. We shall then just set $b = \cup_{\nu<\omega_1} b_\nu$ and be done, of course.

To define b_0, obsere that $\langle c_\alpha \cap \alpha_0 | \alpha < \alpha_0 \rangle$, $S(\alpha_0)$, α_0, $\underset{\sim}{T} | \alpha_0 \in L_{f(\alpha_0)}$ and $L_{f(\alpha_0)} \models$ "α_0, $S(\alpha_0)$, $\underset{\sim}{T} \upharpoonright \alpha_0$ are countable". Hence, working inside $L_{f(\alpha_0)}$, we may let b_0 be the $<_L$-least α_0-branch of $\underset{\sim}{T} \upharpoonright \alpha_0$ such that $b_0 \not\subset U$ for any $U \in S(\alpha_0)$ and $\alpha < \alpha_0 \to b_0 \neq c_\alpha \cap \alpha_0$. Then, since $S(\alpha_0) \subset L_{f(\alpha_0)}$, b_0 has an extension on T_{α_0}.

If $b_0 \subset \ldots \subset b_\nu$ are already (suitably) defined, $b_{\nu+1}$ is defined similarly (to contain the T_{α_ν}-extension of b_ν, of course).

Finally, suppose $\lim(\nu)$ and that $b_0 \subset b_1 \subset \ldots \subset b_\xi \subset \ldots (\xi < \nu)$ are suitably defined. Let $b_\nu = \cup_{\xi<\nu} b_\xi$. Since $\langle \alpha_\nu | \nu < \omega_1 \rangle$ is continuous, b_ν is an α_ν-branch of $\underset{\sim}{T} \upharpoonright \alpha_\nu$ and $\alpha < \alpha_\nu \to b_\nu \neq c_\alpha \cap \alpha_\nu$.

We must show that b_ν extends on T_{α_ν}. We show first that $b_\nu \in L_{f(\alpha_\nu)}$. Clearly, b_ν is definable from $\underset{\sim}{T} \upharpoonright \alpha_\nu$, $\langle \alpha_\xi | \xi < \nu \rangle$, $\langle S(\alpha_\xi) | \xi < \nu \rangle$, $\langle c_\alpha \cap \alpha_\nu | \alpha < \alpha_\nu \rangle$, so it reduces to proving that $\langle \alpha_\xi | \xi < \nu \rangle \in L_{f(\alpha_\nu)}$. And since $\alpha_\xi = \omega_1^{L_{\psi(\xi)}}$ for each ξ, this will be so providing

$\langle L_{\psi(\xi)} | \xi < \nu \rangle \in L_{f(\alpha_\nu)}$. Now, $\psi(\nu) < f(\alpha_\nu)$, so $L_{\psi(\nu)} \in L_{f(\alpha_\nu)}$. Thus, working inside $L_{f(\alpha_\nu)}$, we may define a chain $Y_0 \prec Y_1 \prec \dots \prec Y_\xi \prec \dots \prec L_{\psi(\nu)}$ $(\xi < \nu)$ exactly as $\langle X_\xi | \xi < \omega_1 \rangle$ was defined from L_{ω_2}. But look, in defining $\langle X_\xi | \xi < \nu \rangle$, we could equally well have used X_ν in place of L_{ω_2} (since $\xi < \nu \to X_\xi \prec X_\nu \prec L_{\omega_2}$). Then, since $X_\nu \cong L_{\psi(\nu)}$, an easy induction argument shows that $X_\xi \cong Y_\xi$ for all $\xi < \nu$. Hence $Y_\xi \cong L_{\psi(\xi)}$, $\xi < \nu$. But $\langle Y_\xi | \xi < \nu \rangle \in L_{f(\alpha_\nu)}$, so $\langle L_{\psi(\xi)} | \xi < \nu \rangle \in L_{f(\alpha_\nu)}$, as required.

Thus, b_ν will be proved to extend on level α_ν if we can show that for any $U \in S(\alpha_\nu)$, $b_\nu \not\subset U$. To this end, note first that our above argument did more than prove that $b_\nu \in L_{f(\alpha_\nu)}$. What we actually proved was that b_ν is $L_{\psi(\nu)+1}$-definable (from elements of $L_{\psi(\nu)+1}$.) Now, α_ν is uncountable in $L_{\psi(\nu)+1}$ but countable in $L_{h(\alpha_\nu)+1}$. (For the former, note that $\pi_\nu^{-1} : L_{\psi(\nu)} \prec L_{\omega_2}$ and $\pi_\nu^{-1}(\alpha_\nu) = \omega_1$). Hence $\psi(\nu) < h(\alpha_\nu)$.

<u>Case A:</u> $\psi(\nu) + 1 < h(\alpha_\nu)$.

Then $b_\nu \in L_{h(\alpha_\nu)}$. So as $L_{h(\alpha_\nu)} \models$ "U is Aronszajn" for any $U \in S(\alpha_\nu)$, we cannot have $b_\nu \subset U$ for any such U.

<u>Case B:</u> $\psi(\nu) + 1 = h(\alpha_\nu)$.

Thus $S(\alpha_\nu) \subset L_{\psi(\nu)}$. Let $U \in S(\alpha_\nu)$. For $\iota < \tau \leq \nu$, set $\pi_{\iota\tau} = \pi_\tau \circ \pi_\iota^{-1}$. Then $\langle L_{\psi(\nu)}, (\pi_{\tau\nu})_{\tau<\nu} \rangle$ is (isomorphic to) the direct limit of the elementary system $\langle (L_{\psi(\tau)})_{\tau<\nu}, (\pi_{\iota\tau})_{\iota<\tau<\nu} \rangle$. So for some $\tau < \nu$ and some $U' \in L_{\psi(\tau)}$, $U = \pi_{\tau\nu}(U')$. Let τ be the least such. It is then easily observed that $L_{\psi(\tau)}$ is the smallest $X \prec L_{\psi(\tau)}$ such that $U' \in L_{\psi(\tau)}$ and $X \cap \alpha_\tau$ is transitive. Also, $L_{\psi(\tau)+1} \models$ "α_τ is a regular uncountable cardinal". (By applying π_τ^{-1}).

Thus by lemma 3, it follows that α_τ is countable in $L_{\psi(\tau)+2}$. Hence $\psi(\tau) + 1 = h(\alpha_\tau)$. But look, there is no $L_{\psi(\nu)}$-definable α_ν-branch of U. (By applying π_ν^{-1}). Hence there can be no $L_{\psi(\tau)}$-definable α_τ-branch of U'. Thus $L_{h(\alpha_\tau)} \models$ "U' is Aronszajn". It follows readily that $U' \in S(\alpha_\tau)$. Hence $b_\tau \not\subset U'$. But $U' = U \cap \alpha_\tau$ and $b_\tau = b_\nu \cap \alpha_\tau$. Hence $b_\nu \not\subset U$.

This completes the proof that $\underset{\sim}{T}$ is Kurepa.

Now suppose that $\underset{\sim}{T}$ has an Aronszajn subtree, U. It is easily seen that there is no loss of generality in assuming that U is normal and thin in $\underset{\sim}{T}$. We may also assume that U is the $<_L$-least such. Hence U is L_{ω_2}-definable. So $U \in X_0$. Clearly, $U' = \pi_0(U) = U \cap \underset{\sim}{T}\lceil\alpha_0$. As U is Aronszajn, there is no L_{ω_2}-definable ω_1-branch of U. Thus $L_{\psi(0)+1} \models$ "U' is Aronszajn". But, using lemma 3, it is immediately clear that $\psi(0) + 1 = h(\alpha_0)$. Hence we see that $U' \in S(\alpha_0)$. But by construction, no α_0-branch of U' ever extended on T_{α_0}, so how ever can $U \supset U'$ be cofinal in ω_1? This contradicition completes the proof. ∎

The above theorem, together with our next result, shows that $V = L \to \neg\Phi$.

Theorem 5

Assume GCH. Suppose that there is a Kurepa tree $\underset{\sim}{T}$, no subset of which is an Aronszajn tree. Then $\neg\Phi$.

Proof: Without loss of generality, we may assume that $T \subset \bigcup_{\alpha<\omega_1} 2^\alpha$ and that $s \leq_T t$ iff s is an initial segment of t (written s inl t).

Let ρ be the set of all ω_1-branches of $\underset{\sim}{T}$, ordered lexicographically. Thus $|\rho| = \omega_2$. We show that $\omega_2, \omega_2^* \not\leq \rho$.

Let $D = \{s \in \cup_{\alpha<\omega_1} 2^{\alpha} | (\exists b \in \rho)(s \ \underline{\text{inl}} \ b)\}$. Then $|D| = \omega_1$ by GCH .

Suppose $\omega_2 \leq \rho$, and let $\langle b_\nu | \nu < \omega_2 \rangle$ be an ω_2-sequence from ρ . Let $A = \{\nu \in \omega_2 | \lim(\nu)\}$. By induction on $\nu \in A$, pick $s_\nu \in D$ such that $\tau \in A \cap \nu \to s_\tau \neq s_\nu$ (By demanding that $s_\nu \ \underline{\text{inl}} \ b_\nu$ but $\neg s_\nu \ \underline{\text{inl}} \ b_{\nu+1}$ for each $\nu \in A$) . Then $\{s_\nu | \nu \in A\}$ is a set of ω_2 distinct members of D, which is absurd. Similarly if $\omega_2^* \leq \rho$.

Now let $\sigma \subset \rho$, $|\sigma| = \omega_1$. We show that $\omega_1 \leq \sigma$ or $\omega_1^* \leq \sigma$. Suppose $\rho = \langle \rho, \dashv \rangle$.

Let $U = \{s \in \cup_{\alpha<\omega_1} 2^{\alpha} | (\exists b \in \sigma)(s \ \underline{\text{inl}} \ b)\}$. Thus $U \subset \underset{\sim}{T}$.

Case 1: For some $b \in \sigma$ it is the case that whenever $s \ \underline{\text{inl}} \ b$ there is $b' \in \sigma$ such that $s \ \underline{\text{inl}} \ b'$ and $b' \neq b$. By induction, pick a strictly $<_T$-increasing sequence $\langle s_\nu | \nu < \omega_1 \rangle$ of initial sections of b , and a pairwise distinct sequence $\langle b_\nu | \nu < \omega_1 \rangle$ of members of $\sigma - \{b\}$ such that $s_\nu \ \underline{\text{inl}} \ b_\nu$.

Then either $\{b_\nu | \nu < \omega_1 \ \& \ b_\nu \dashv b\}$ or else $\{b_\nu | \nu < \omega_1 \ \& \ b \dashv b_\nu\}$ has cardinality ω_1 . But $\langle b_\nu | \nu < \omega_1 \ \& \ b_\nu \dashv b \rangle$ is a $\dashv$ -increasing sequence from σ and $\langle b_\nu | \nu < \omega_1 \ \& \ b \dashv b_\nu \rangle$ is a $\dashv$-decreasing sequence from σ .

Case 2: Otherwise.

Then, for each $b \in \sigma$ there is $s_b \in U$ such that $s_b \ \underline{\text{inl}} \ b$ and for all $b' \in \sigma$, $s_b \ \underline{\text{inl}} \ b' \to b' = b$.
Let $U' = \{s | (\exists b \in \sigma)(s \ \underline{\text{inl}} \ s_b)\}$. Then $U' \subset U$. We know that U' cannot be an Aronszajn tree. And yet U' is an (ω_1, ω_1)-tree. Hence U' has an ω_1-branch, d . For $b_1, b_2 \in \sigma$, s_{b_1} and s_{b_2} must be $<_T$-incomparable. Hence for each $s \ \underline{\text{inl}} \ d$ there must be a $b \in \sigma$ such that $s \ \underline{\text{inl}} \ s_b$

and $\neg s_b$ <u>inl</u> d . So, for each s <u>inl</u> d there is $b \in \sigma$ such that s <u>inl</u> b and $b \neq d$. So, as in Case 1, $\omega_1 \leq \sigma$ or $\omega_1^* \leq \sigma$. ∎

§3. Subtrees of ω_1-trees.

We saw above that it is <u>consistent</u> that there is a Kurepa tree with no Aronszajn subtree. Since the existence of Kurepa trees is not provable in ZFC (see [4]), we could not hope to eliminate the use of V = L in establishing that result. However, in ZFC, it <u>is</u> possible to construct a normal (ω_1, ω_1)-tree with no Aronszajn subtree. In fact, we have:

Theorem 6.

there is a normal (ω_1, ω_1)-tree $\underset{\sim}{T}$ such that:

(i) $\underset{\sim}{T}$ has no Aronszajn subtree.

(ii) if $\underset{\sim}{T}'$ is any normal (ω_1, ω_1)-tree then either $\underset{\sim}{T}'$ has an Aronszajn subtree on else $\underset{\sim}{T} \leq \underset{\sim}{T}'$.

<u>Proof</u>: Let $T = \{s \in 2^{\omega_1} \mid |\{\alpha \in \omega_1 | s(\alpha) = 1\}| < \omega\}$, and make T into a tree by setting $s \leq t$ iff $s \subseteq t$. Clearly, $\underset{\sim}{T}$ is a normal (ω_1, ω_1)-tree such that every point in T lies on an ω_1-branch of $\underset{\sim}{T}$.

(i) Suppose $U \subseteq T$, U a normal (ω_1, ω_1)tree. We show that U is not an Aronszajn tree. It is easily seen that we may assume that U is an initial segment of $\underset{\sim}{T}$. (If the initialisation of U in $\underset{\sim}{T}$ has an ω_1-branch, so must U itself !) Set $C = \{\alpha \in \omega_1 | \lim (\alpha)\}$. For each $\alpha \in C$, let $s_\alpha \in U$ be arbitrary. For each $\alpha \in C$, define $f(\alpha)$ = the largest $\beta < \alpha$ such that $s_\alpha(\beta) = 1$, or else $f(\alpha) = 0$ if no such β exists. Then $f: C \to \omega_1$ is regressive, so by a well known theorem of Fodor (see [1], Chapter 3, for example) we can find stationary set $X \subseteq C$ such that $f''X = \{\beta_0\}$ for some fixed β_0. It follows immediately that there must be an uncountable set $Y \subseteq X$ such that $\alpha, \beta \in Y \& \alpha < \beta \to s_\alpha \leq_T s_\beta$. Hence $\{s_\alpha | \alpha \in Y\}$ determines an ω_1-branch of U.

(ii) Let $\underset{\sim}{T}'$ be a normal (ω_1, ω_1)-tree with no Aronszajn subtree. By replaing $\underset{\sim}{T}'$ by a subtree if necessary, we assume $\underset{\sim}{T}'$ is such that every point in $\underset{\sim}{T}'$ has exactly two distinct immediate successors. But look, as $\underset{\sim}{T}'$ has no Aronszajn subtree, every point of $\underset{\sim}{T}'$ lies on an ω_1-branch of $\underset{\sim}{T}'$. It is now an easy matter to inductively (on the levels) embed $\underset{\sim}{T}$ into $\underset{\sim}{T}'$. ∎

Let us return now to the property Φ, and show how the failure of Φ is closely connected with the existence of Aronszajn subtrees of trees. Let Δ denote the following proposition, often refered to as <u>Chang's Conjecture</u>: If $\mathfrak{A} = < \omega_2, \omega_1, \ldots >$ is an arbitrary first-order structure with a countable language, there is $\mathfrak{B} = < B, B \cap \omega_1, \ldots > \prec \mathfrak{A}$ such that $|B| = \omega_1$ and $|B \cap \omega_1| \leq \omega$. Silver [3] has shown that Con (ZFC = "there is a Ramsey cardinal") $\to$ Can (ZFC + GCH + Δ). (And since Δ implies the existence of Solovay's $0^{\#}$, the large cardinal assumption here probably cannot be weakened very much, if at all.)

<u>Theorem 7</u>.

Assume Δ + GCH. If Φ fails, then there is an ω_2-Aronszajn tree with no Aronszajn subtree.

<u>Proof</u>: Let ρ be an order type of cardinality ω_2 such that $\omega_2, \omega_2^* \not\leq \rho$, but for any $\sigma \subseteq \rho$ of cardinality ω_1, either $\omega_1 \leq \sigma$ or else $\omega_1^* \leq \sigma$. Assume $\rho = \langle \omega_2, -3 \rangle$ for definiteness.

Define, by induction on the levels, a tree, $\underset{\sim}{T}$, as follows: The elements of $\underset{\sim}{T}$ are non-empty intervals of ρ and the ordering, $<_T$, is $\supset$. Set $T_0 = \{\rho\}$. If $I \in T_\alpha$ and $|I| = 1$, I has no successors in $\underset{\sim}{T}$. If $I \in T_\alpha$ and $|I| > 1$, let α_I be the least ordinal in I not maximal in I and let $\{\xi \in I | \xi \underline{\prec} \alpha_I\}$ and $\{\xi \in I | \alpha_I \prec \xi\}$ be the successors of I in $T_{\alpha+1}$. Finally, if $\lim(\delta)$ and $\underset{\sim}{T}\restriction\delta$ is defined, let $T_\delta = \{\cap b | b$ is a δ-branch of $T\restriction\delta$ and $\cap b \neq \emptyset\}$.

We linearly order each level T_α by setting, for each $I, J \in T_\alpha$, $I <_\alpha J$ iff $(\forall\xi \in I)(\forall\zeta \in J)(\xi \prec \zeta)$.
Suppose that for some $\alpha < \omega_2$, $|T_\alpha| = \omega_2$.

Let α be the least such. Then, clearly, $\lim(\alpha)$.
Let $f\colon \underset{\sim}{T}\restriction \alpha + 1 \leftrightarrow T_\alpha$, and consider the structure

$\mathfrak{A} = \langle \underset{\sim}{T}\restriction\alpha + 1, T\restriction\alpha , T_\alpha , f, <_\alpha , <^*\rangle$, where $<^*$ is the lexicographic order on $\underset{\sim}{T}\restriction\alpha + 1$ induced by the $<_\beta$, $\beta < \alpha$.
Now, $|\underset{\sim}{T}\restriction\alpha + 1| = \omega_2$ and $|T\restriction\alpha| = \omega_1$ (by choice of α), so let $\mathcal{L} = \langle X, X \cap T\restriction\alpha, \ldots\rangle \prec \mathfrak{A}$ with $|X| = \omega_1$ and $|X \cap T\restriction\alpha| = \omega$, by assumption. Since

(i) $\mathfrak{A} \models (\forall x, y \in T_\alpha)(x <_\alpha y \rightarrow (\exists z \in T\restriction\alpha)(x \leq^* z \leq^* y))$

it follows that

(ii) $\mathcal{L} \models (\forall x, y \in T_\alpha \cap X)(x <_\alpha y \rightarrow (\exists z \in T\restriction\alpha \cap X)(x \leq^* z \leq^* y))$

For each $I \in T_\alpha \cap X$, let $\theta(I) \in I$ be arbitrary.
Then, for $I, J \in T_\alpha \cap X$, $I <_\alpha J$ iff $\theta(I) \prec \theta(J)$.

Set $\sigma = \{\theta(I) | I \in T_\alpha \cap X\}$. Thus $\sigma \subset \rho$. And since $f\colon X \leftrightarrow T_\alpha \cap X$, $|\sigma| = \omega_1$. We show that $\omega_1, \omega_1^* \not\leq \sigma$, a contradiction. Suppose $\omega_1 \leq \sigma$. (The case $\omega_1^* \leq \sigma$ is similar). It follows that $\omega_1 \leq \langle T_\alpha \cap X, <_\alpha \cap X^2\rangle$, so let $\langle I_\nu | \nu < \omega_1\rangle$ be a $<_\alpha$increasing sequence from $T_\alpha \cap X$. By (ii), we can inductively pick mimbers J_ν from $T\restriction\alpha \cap X$, for $\nu = \cup\nu < \omega_1$, so that $\lim(\nu_1)$ & $\lim(\nu_2)$ & $\nu_1 < \nu_2 \rightarrow I_{\nu_1} <^* J_{\nu_1} <^* I_{\nu_2}$. But $|T\restriction\alpha \cap X| = \omega$, so we have a contradicition.

Hence, for all $\alpha < \omega_2$, $|T_\alpha| \leq \omega_1$.

Suppose $\underset{\sim}{T}$ has an ω_2-branch $b = \langle I_\alpha | \alpha < \omega_2 \rangle$. For each $\alpha < \omega_2$ there is $J_{\alpha+1} \in T_{\alpha+1}$ such that $I_\alpha <_T J_{\alpha+1}$ and $J_{\alpha+1} \neq I_{\alpha+1}$. Let $\theta_{\alpha+1} \in J_{\alpha+1}$ for each α . Either $|\{\theta_{\alpha+1} | \alpha < \omega_2 \ \& \ I_{\alpha+1} <_{\alpha+1} J_{\alpha+1}\}| = \omega_2$ or else $|\{\theta_{\alpha+1} | \alpha < \omega_2 \ \& \ J_{\alpha+1} <_{\alpha+1} I_{\alpha+1}\}| = \omega_2$. In the first case, the requisite $\theta_{\alpha+1}$'s form a $\dashv$-decreasing chain of type ω_2 , in the second case it is $\dashv$ -increasing. So, in either event, we have a contradiction, since $\omega_2, \omega_2^* \not\leq \rho$. Hence $\underset{\sim}{T}$ is an ω_2-Aronszajn tree.

Clearly, any Aronszajn subtree of $\underset{\sim}{T}$ will likewise correspond to a subtype $\sigma \subseteq \rho$, $|\sigma| = \omega_1$, such that $\omega_1, \omega_1^* \not\leq \sigma$. Thus $\underset{\sim}{T}$ cannot have an Aronszajn subtree, and we are done. ∎

Remark. By a simple generalisation of the proof of Theorem 6, one can construct, in ZFC, a normal (ω_2, ω_2)-tree with no Aronszajn (and no ω_2-Aronszajn) subtree. We do not know if it is possible to construct, in ZFC + GCH, an ω_2-Aronszajn tree with no Aronszajn subtree. If such were possible, however, we would immediately have a proof of $\neg \Phi$ in ZFC + GCH , since the lexicographic ordering of such a tree is easily seen to provide a counterexample to Φ . In view of an last result, this would seem to be the only hope of establishing $\neg \Phi$ in ZFC + GCH . However, by Δ itself, the more obvious sorts of ω_2-Aronszajn trees which one can construct in ZFC + GCH do have Aronszajn subtrees, so this approach does not appear to be very hopeful. Much more likely, in our opinion, is that in Silver's model of Δ (or perhaps a slight modification of it), every ω_2-Aronszajn tree does have an Aronszajn subtree, whence, by theorem 7, we have at once the consistency of ZFC + GCH + Φ . Unfortunately, a proof of this has so far eluded us.

Postscript

Since writing this paper, we obtained a proof of the result Con (ZFC + "there is a Ramsey cardinal") $\rightarrow$ Con (ZFC + $2^{\omega}=2^{\omega_1}=\omega_2$ + Φ) . The proof will appear elsewhere. We still do not know if Φ is consistent with GCH .

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